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Quantum anharmonic symmetrical oscillators using elliptic functions

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Abstract. We study in the JWKB approximation the energy levels of the symmetric anharmonic oscillators $V(x) = Ax^2 + Bx^4$ for different signs and values of A and B. Comparisons are made with published results for specific cases and with numerical calculations. We give an additional example of exact value, to add to the very rare catalogue of known examples.

1. Introduction

In a series of papers we have studied complete analytical solutions of different anharmonic oscillators. Díaz Bejarano *et al* (1982) have considered classical oscillators in potentials of the form $V(x) = Ax^2 + Bx^4$; the more difficult case of the asymmetrical potential $V(x) = Ax^2 + Bx^3$ has been considered by Díaz Bejarano *et al* (1985). Finally we show the relation between the relativistic oscillators and the non-linear symmetrical oscillator (Díaz Bejarano and Martín Sánchez 1983). The intrinsic non-linear properties of the relativistic oscillator make a general study of the anharmonic symmetrical oscillator in quantum mechanics interesting. These oscillators are also important in chemistry and have many applications†. The case with $A = +1$ and B positive has been treated by many different methods (Bazley and Fox 1961, Reid 1965, Bender and Wu 1969, Laane 1970, Loeffel *et al* 1969, Pascual 1969, Biswas *et al* 1971, Mathews and Eswaran 1972, Banerjee 1976, Hioe and Montroll 1975, Banerjee *et al* 1978, Hioe *et al* 1978, Radmore 1980, Mathews *et al* 1981, Cizek and Vrscay 1982, Hirsbrunner 1982, Fernández and Castro 1983, Fernández *et al* 1983, Flessas 1983, Flessas *et al* 1983, Bhattacharya *et al* 1984, Marziani 1984).

2. Eigenvalues from the JWKB approximation for the anharmonic symmetrical oscillators

For the calculation of the energy levels we use the Bohr-Sommerfeld rule modified by the JWKB method, in the lowest order (Galindo and Pascual 1978) i.e.

$$\oint p(x)\,\mathrm{d}x = (n+\tfrac{1}{2})h. \tag{1}$$

Taking into consideration the symmetry of the problem and changing to the more

† A list of references (320) for previous work can be found, for example in Killingbeck (1977). More references can be found in Martín Sánchez (1983).

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convenient t variable (Bradbury 1968) we transform (1) into

$$4\mu \int_0^{T/4} \dot{x}^2(t)\, dt = (n+\tfrac{1}{2})h. \tag{2}$$

We give in table 1 the corresponding classical periodic solutions for general coefficients A and B in the potential

$$V(x) = Ax^2 + Bx^4.$$

In this table, m is the parameter of the elliptic functions and m_1 is the complementary parameter (Abramowitz and Stegun 1972); x_0 is the maximum amplitude for the oscillations and $\dot{x}_0$ is the maximum velocity.

We study here the following cases: (α) $A>0$, $B>0$; (β) $A>0$, $B<0$; (γ) $A<0$, $B>0$. The type (δ) $A<0$, $B<0$ has no bound energy levels.

For the type α there are various special cases. For $m=0$, $B=0$ we have the limit corresponding to the harmonic oscillator. The solutions cn and sd are the well known equivalent solutions cos and sin (Abramowitz and Stegun 1972). For $m=\frac{1}{2}$, $A=0$ and we have the limit of the quartic oscillator, to be considered in § 5. $A=B=0$ is the free particle and is not studied here.

The energy levels for the type β must lie below the potential maximum; they are resonances because there is tunnelling through the barriers.

For the type γ when $E=0$ the four classical solutions (cn, sd, dn and nd) are equivalent and $m=1$.

Some aspects and formulae for the case with $A=+1$ and B negative are also known (Biswas *et al* 1973, Drummond 1981). Numerical solutions for special values of A negative and B positive are given by Somorjai and Hornig (1962), Caswell (1979), Dias de Deus (1982) and Balsa *et al* (1983). From the enormous literature on the subject we do not know of any general discussions of the problem. In this paper we intend to make this general study. The results have been checked with results previously published (mentioned above) and/or with calculations made with numerical methods (Oset and Salcedo 1984).

For the Hamiltonian

$$H = (P^2/2\mu) + AX^2 + BX^4 \tag{3}$$

μ is the mass of the oscillator and P and X momentum and position. Díaz Bejarano *et al* (1983) gave the classical solutions, needed for the application of the JWKB method, obtained from the applications of Newton's law, i.e. with coefficients in the expression for the force.

We also give in table 1 the classical total energy E of the particle (kinetic + potential). The total energy of the particle is then, according to table 1, for type α

$$E = Ax_0^2 + Bx_0^4 = m_1\mu\omega^2 x_0^2/2. \tag{4}$$

From (3) and (2) we obtain

$$4\mu x_0^2\omega^2 \int_0^{T/4} \mathrm{sn}^2\,\omega t\, \mathrm{dn}^2\,\omega t\, dt = (n+\tfrac{1}{2})h. \tag{5}$$

The period of the cn ωt is $4K/\omega$, where K is the complete elliptic integral of the first kind (Abramowitz and Stegun 1972). The integral (5) and similar integrals for the

Table 1. Classical equations of motion for non-linear oscillators. Potential type $V(x) = Ax^2 + Bx^4$. Initial conditions are: I: $x(0) = x_0$, $\dot{x}(0) = 0$; II: $x(0) = 0$, $\dot{x}(0) = \dot{x}_0$.

Potential type	Initial conditions	Classical solutions	ω^2	x_0^2 or $\dot{x}_0^2$	Total energy	Condition for m
α	I	$x_0 \operatorname{cn} \omega t$	$2A[\mu(1-2m)]^{-1}$	$mA[(1-2m)B]^{-1}$	$mm_1A^2[(1-2m)^2B]^{-1}$	$0 \leqslant m \leqslant \frac{1}{2}$
	II	$(\dot{x}_0/\omega) \operatorname{sd} \omega t$		$2mm_1A^2[(1-2m)^2\mu B]^{-1}$		
β	I	$x_0 \operatorname{cd} \omega t$	$2A[\mu(1+m)]^{-1}$	$-mA[(1+m)B]^{-1}$	$-mA^2[(1+m)^2B]^{-1}$	$0 \leqslant m \leqslant 1$
	II	$(\dot{x}_0/\omega) \operatorname{sn} \omega t$		$-2mA[(1+m)\mu B]^{-1}$		
		As case α for $E \geqslant 0$, but $\frac{1}{2} \leqslant m \leqslant 1$				
γ	I	$x_0 \operatorname{dn} \omega t \quad E < 0$	$2A[\mu(m-2)]^{-1}$	$A[(m-2)B]^{-1}$	$-m_1A^2[(m-2)^2B]^{-1}$	$0 \leqslant m \leqslant 1$
	I	$x_0 \operatorname{nd} \omega t$		$m_1A[(m-2)B]^{-1}$		

other types are tabulated in Byrd and Friedman (1954). After some simple algebra one obtains for the energy level n in the potential type α

$$E_n = (n+\tfrac{1}{2})h\omega 3mm_1\{8[(2m-1)E(K)+m_1K]\}^{-1} \tag{6}$$

where $E(K)$ is the complete elliptic integral of the second kind.

For initial conditions of type II, the classical equations of motion for the same potential is given in table 1 and the total energy for these initial conditions can also be seen in this table. The JWKB quantisation condition can now be applied, and after minor algebra the same value for E_n as before is obtained. This result is the expected one for equivalent solutions of the same potential. To be able to calculate the energy levels from (6) one needs a relation between ω, m, A and B that can be obtained from the values given in table 1.

Similar formulae can be found for all the potentials. Then, making the same operations as indicated above we obtain for the potential type β, the energy

$$E_n = (n+\tfrac{1}{2})h\omega 3m\{8[(1+m)E(K)-m_1K)]\}^{-1}$$

but, in these cases, the energies given by the preceding formula are not energy levels; they are only resonances because a particle in the well of potential β can traverse the limiting barriers by tunnelling. We discuss this effect in the next section; there, we calculate also the resonance widths for the corresponding energy.

The positive energy levels of potential γ are the same as for case α, but the restriction on the parameter of elliptic function is now $\frac{1}{2} < m < 1$, because the A value must be in agreement with the values of ω and E in table 1. For the calculations of the negative energy levels of type γ it is necessary to calculate the tunnelling effect of the intermediate barrier between the two potential wells. This effect will be considered in the next section.

When we calculate the negative energy levels we must check the connection with the positive ones. This will be made in the last section giving the numerical values for both positive and negative energy levels.

We need only calculate the energy levels with $A=1$ whatever B is, or with $B=1$ whatever A is, because we can use the scaling relations satisfied by the exact eigenvalues

$$E_n(A, B) = A^{1/2}E_n(1, BA^{-3/2})$$

$$E_n(A, B) = B^{1/3}E_n(AB^{-2/3}, 1)$$

or the relation obtained with the two preceding

$$A^{1/2}E_n(1, BA^{-3/2}) = B^{1/3}E_n(AB^{-2/3}, 1).$$

3. Potential barriers

In the previous calculations it has not been necessary to find the relations between the parameter of the elliptic functions and the turning points. Now, for the calculus of barrier penetration we calculate the solutions of the energy conservation equation

$$V(x) = E. \tag{7}$$

The roots of this equation are the turning points. Another relation is given in table 1 for the total energy. Comparing the two relations for the m value we have the equality

$$m = a^2/(a^2+|\alpha|^2)$$

where a and α are the roots of (7). The other roots of this equation are $-a$ and α^* taking into consideration the symmetry of the potential. The complementary parameter

is

$$m_1 = |\alpha|^2/(a^2+|\alpha|^2).$$

Similarly for the other potentials (β and γ) we obtain the following relations.

For type β

$$m = a^2/b^2, \qquad m_1 = 1 - a^2/b^2.$$

For type γ and negative energies (for positive energies it has the same relation as potential α)

$$m = 1 - a^2/b^2, \qquad m_1 = a^2/b^2.$$

The roots of equation (7) in both cases are a, $-a$, b and $-b$. The similarity between the two types is evident and it will be used in the following. Another important property of the potential β and γ is that $V(x)$ in the type β is $-V(x)$ in the type γ and vice versa.

From the last property and the property of the elliptic function of having two periods, it follows that the elliptic functions are the 'imaginary classical solutions' for all these potentials. This is seen from the expression for the classical oscillation period

$$T = \nu^{-1}(E_n) = 2\mu \int_a^b p^{-1}\,\mathrm{d}x = 2\int_a^b \{2[E - V(x)]/\mu\}^{-1/2}\,\mathrm{d}x$$

(a and b are two adjacent turning points). We use the last formula in all the cases, including when $V(x) > E$ (imaginary solutions).

To calculate the widths of the resonances in the potential type β we use the expression (Shepard 1983).

$$\Gamma(E_n) = 2\hbar\nu(E_n)\exp(-2K_{II})$$

taking into consideration the symmetry of the potential. The value of K_{II} is

$$K_{II} = \hbar^{-1}\int_a^b |p|\,\mathrm{d}x.$$

Then following Byrd and Friedman (1954), we obtain

$$K_{II} = (-2E_n/3m_1\hbar\omega)[(2-m)E(K) - 2m_1K].$$

We change $m \leftrightarrow m_1$ and insert the values of E_n calculated in the previous paragraph to obtain

$$K_{II} = (2E_n/3m\hbar\omega)[(1+m)E' - 2mK'].$$

K' and E' are the complementary elliptic integrals of the first and the second kind respectively.

The same results for $A = 1$ have been obtained (without using the varaible t) by Shepard (1983). This author does not give numerical calculations in his work; our numerical values are given in the last section of this paper.

We can calculate the period for this case, $T = 4K'/\omega$. The real period is $4K/\omega$.

4. Double well potential

In the calculation of energy levels in the potential type γ if $E<0$ there is a potential barrier between two potential wells; we must take into account the splitting of degenerate energy levels due to tunnelling. Then, the energy levels are

$$E_n = E_n^{(0)} \pm \Delta E_n.$$

Here $E_n^{(0)}$ is the value obtained using the quantification rule only in a well of the potential. ΔE_n is the term of tunnelling between the two wells; this term is responsible for the splitting in energy for every double degenerate level.

Using the same procedure to obtain the quantified levels as in § 2, we arrive at the following result:

$$E_n^{(0)} = (n+\tfrac{1}{2})\hbar\omega 3m_1\{4[(m-2)E(K)+2m_1K]\}^{-1}.$$

For the expression of ΔE_n, we use the considerations of the preceding section. The results are

$$\Delta E_n = (\hbar\omega/2K')\exp\{(-4E_n^{(0)}/3m_1\hbar\omega)[(2-m)E'-mK']\}.$$

Here ω, A, B and E are related via the values given in table 1. Banerjee *et al* (1978) have made a similar study and our results are in very good agreement. For the numerical calculations these authors use another method. Comparisons are made in the next section.

5. Comparisons with well known limits

For the potential type α the case

$$m=0, \qquad A=\tfrac{1}{2}\mu\omega^2, \qquad B=0$$

is the limiting case corresponding to the harmonic oscillator. For $m=0$

$$\mathrm{sn}(\omega t;0)=\mathrm{sen}\ \omega t, \qquad \mathrm{dn}(\omega t;0)=1, \qquad T=2\pi/\omega, \qquad E=\mu x_0^2\omega^2/2$$

(see Abramowitz and Stegun 1972) and from (2) one obtains the well known formula for the energy levels of the harmonic oscillator.

The case

$$m=m_1=\tfrac{1}{2}, \qquad A=0, \qquad B=\mu\omega^2/4x_0^2=\mu\omega^4/8\dot{x}_0^2$$

is the limiting case corresponding to the quartic oscillator†. To compare with published results we restrict ourselves to the case $B=1.0$ and we use, as did Galindo and Pascual (1978), the units $\hbar=2\mu=1$. Then the energy levels are

$$E_n=\{(n+\tfrac{1}{2})3\pi[K(m=\tfrac{1}{2})]^{-1}\}^{4/3}.$$

This result is identical with the usual

$$E_n=[(n+\tfrac{1}{2})\pi^{1/2}4\Gamma(\tfrac{3}{2}+\tfrac{1}{4})/\Gamma(\tfrac{1}{4})]^{4/3}$$

where Γ is the Euler gamma function. The equality can be proved using (see the appendix)

$$2^{-1/2}K(\tfrac{1}{2})=(\tfrac{1}{4})B(\tfrac{1}{4},\tfrac{1}{2})=(\tfrac{1}{4})[\Gamma(\tfrac{1}{4})\Gamma(\tfrac{1}{2})]/\Gamma(\tfrac{1}{4}+\tfrac{1}{2})$$

where B is the beta function. The energy levels are also in agreement with the results of Bender *et al* (1977).

† For a general review of the JWKB method including the case of the quartic oscillator see Voros (1983).

For the type β the case

$$m = 0, \qquad m_1 = 1, \qquad A = \mu\omega^2/2, \qquad B = 0$$

is once more the limiting case corresponding to the harmonic oscillator, with the same conclusions as those expressed above.

Now we turn to the study of the numerical values of the energy levels. The well type α has been frequently studied (see the references of § 1). It is usual to express the results in units of the fundamental level of the harmonic oscillator. Alternatively one can use the units $\hbar = 2\mu = 1$. Our results for the first five energy levels are given in figure 1 for $A = 1.0$. We compare the JWKB results in the lower order with values obtained with numerical integration methods (Oset and Salcedo 1984) and/or from the very extensive literature on the subject (see § 1). The values for variable A can be calculated from the scaling relations if necessary. We can observe in figure 1 the good agreement between our calculations and more sophisticated methods, excluding the ground level, because it is well known that the JWKB approximation is worse the lower the energy value is, particularly for the ground energy level.

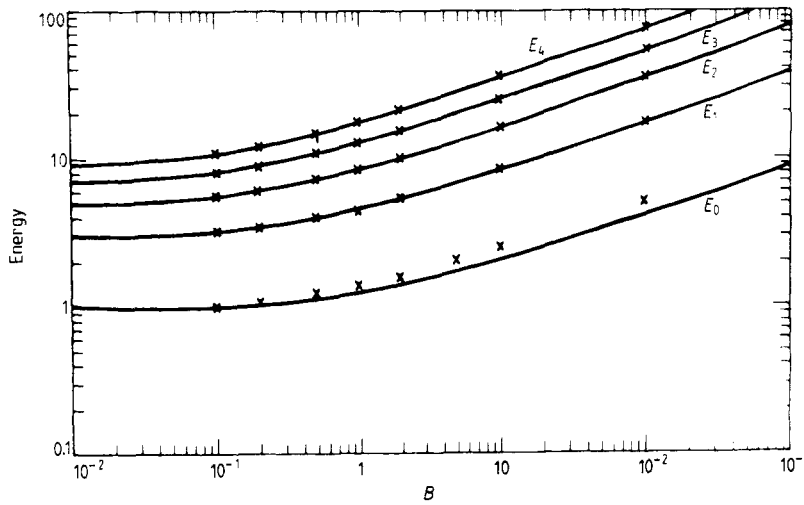

Figure 1. Energy levels of the oscillator $V(x) = x^2 + Bx^4$. The curves are our results for the energy levels. The crosses are values found by other authors (see references in § 1).

A note to add to the results is that the energy level values calculated by the JWKB approximation are below those calculated by other methods for every value of B due to the closeness of this potential compared with the harmonic oscillator potential, which is exact in the JWKB approximation.

We have indicated in § 3 an analytic comparison of the equations for the potential type β. We can now compare with numerical values. We plot in figure 2 the first five values of the resonances, compared with those calculated by Drummond (1981) using perturbative methods. For some cases table 2 shows this comparison. We can observe in figure 2 and table 2 that the JWKB energy levels are above those calculated by Drummond. The reason is the expansion of the potential well in comparison with the harmonic oscillator potential. The values for the widths of the first five resonances are shown in figure 3 and table 3. The comparison has been made taking into

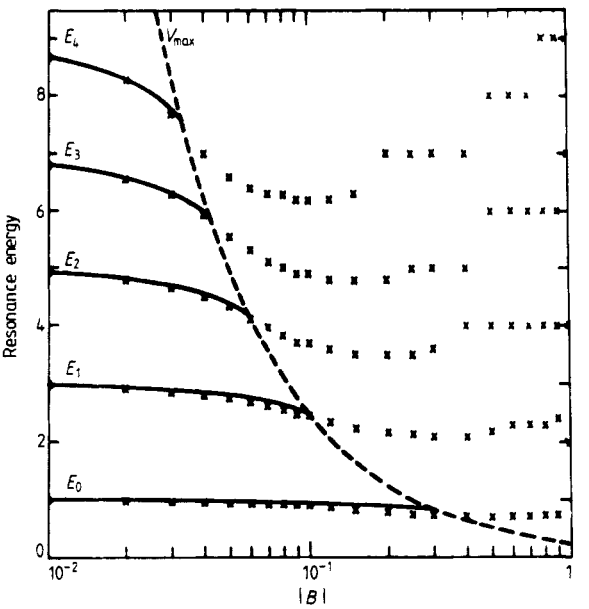

Figure 2. 'Resonances' of the oscillator $V(x) = x^2 - Bx^4$. The full curves are our results for the energy levels. The crosses are Drummond's values (1981). The broken curve is the maximum potential value.

consideration the relation (Shepard 1983)

$$\Gamma(E_n) = 2\,\mathrm{Im}(E_n).$$

$\mathrm{Im}(E_n)$ is the imaginary part of the energy. The discrepancy in the widths is justified by the limitations of the JWKB method: it is poor for low and deeply bound levels (Galindo and Pascual 1978), but the agreement is good in general, taking into account the low order used in the approximation. Therefore, the JWKB values are at first below Drummond's but finally it is observed that the opposite effect occurs. In conclusion, it must be an exact value obtained by the JWKB method for the resonance widths. This subject will be discussed for the case γ.

We now study the potential type γ, taking tunnelling effects into consideration. The values of the energy levels and the comparison with the Banerjee and Bhatnagar (1978) and the Somorjai and Horning (1962) values are plotted in figure 4 for $A = -1.0$. The values of this figure are

$$E'_n = E_n + \tfrac{1}{4}B$$

i.e. all the values are greater than zero because of the logarithmic representation, the potential minimum $\frac{1}{4}B$ having been added. Some results for common cases are given in table 4. We have already noted the important fact that the JWKB energy levels appear above the exact values whereas for large B ($A = 1$ for example) the JWKB values are below the exact ones. This effect occurs because those energy levels between the maximum and minimum of the potential behave as potential type β due to transmission across the barrier, but above the maximum the behaviour is like the simple well (potential type α). Then the graphics lines of exact and approximate values must

Table 2. Comparison of the values of the resonance energies for the oscillator $V(x) = x^2 - Bx^4$; $B > 0$ with the values of Drummond (1981).

	E_0		E_1		E_2		E_3		E_4		
B	WKB	Ref	WKB	Ref	WKB	Ref	WKB	Ref	WKB	Ref	V_{max}
0.000	1.000	1.000	3.000	3.000	5.000	5.000	7.000	7.000	9.000	9.000	—
0.001	1.000	—	2.997	—	4.992	—	6.982	—	8.970	—	250.000
0.010	0.996	0.992	2.966	2.961	4.903	4.898	6.806	6.801	8.674	8.669	25.000
0.020	0.992	0.984	2.930	2.920	4.797	4.786	6.586	6.574	8.285	8.269	12.500
0.030	0.989	0.976	2.891	2.876	4.680	4.659	6.327	6.291	7.767	7.698	8.333
0.040	0.985	0.967	2.851	2.827	4.548	4.505	5.982	5.91	—	7.05	6.250
0.050	0.981	0.958	2.808	2.771	4.391	4.315	—	5.54	—	6.6	5.000
0.060	0.976	0.948	2.762	2.707	4.166	4.12	—	5.3	—	6.4	4.167
0.070	0.972	0.937	2.711	2.636	—	3.96	—	5.1	—	6.3	3.571
0.080	0.968	0.926	2.655	2.556	—	3.83	—	5.0	—	6.3	3.125
0.090	0.964	0.913	2.589	2.50	—	3.7	—	4.9	—	6.2	2.778
0.100	0.959	0.901	2.500	2.45	—	3.7	—	4.9	—	6.2	2.500

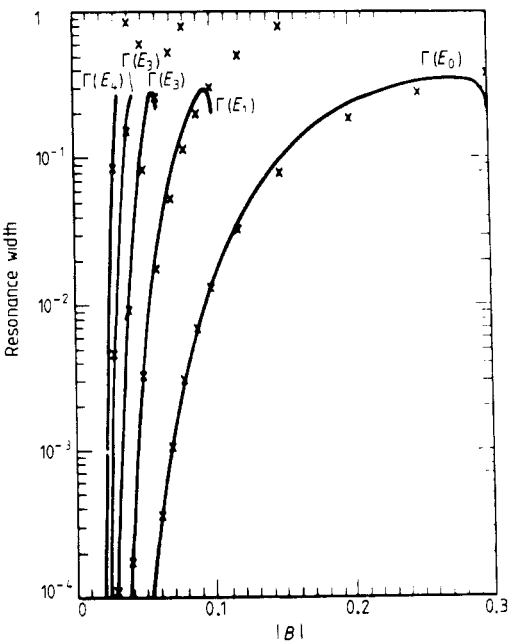

Figure 3. Widths of the resonances of the oscillator $V(x) = x^2 - Bx^4$. The curves are our results. The crosses are Drummond's values (1981).

intersect each other. This means that the approximate energy levels calculated by the JWKB approximation are exact even in the lower order of this approximation. It is an unusual phenomenon. In table 4, two examples can be seen: for $B = 0.130$, the level E_3, or for $B = 0.75$, the level E_1; they are always odd levels. The effect is complete

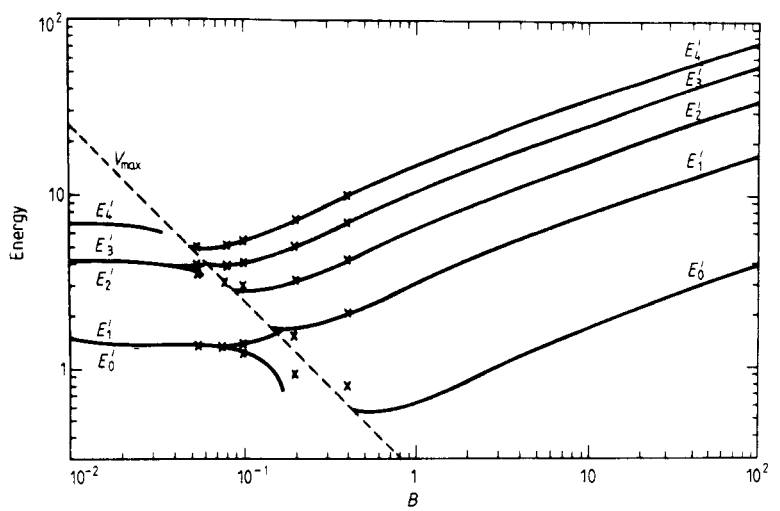

Figure 4. Energy levels of the oscillator $V(x) = -x^2 + Bx^4$. The full curves are our results for the energy levels ($E_n' = E_n + \frac{1}{4}B$). The crosses are the Somorjai and Hornig (1961) and Banerjee and Bhatnagar (1978) values. The broken line is the value of the maximum potential ($V_{max} = \frac{1}{4}B$).

Table 3. Comparison of the values of $\Gamma(E_n)$ with Drummond's values (1981) for the potential $V(x) = x^2 - Bx^4$; $B > 0$. The numbers in parentheses are negative powers of ten.

	$\Gamma(E_0)$		$\Gamma(E_1)$		$\Gamma(E_2)$		$\Gamma(E_3)$		$\Gamma(E_4)$		
B	WKB	Ref	WKB	Ref	WKB	Ref	WKB	Ref	WKB	Ref	V_{max}
0.000	0.000	—	0.000	—	0.000	—	0.000	—	0.000	—	—
0.001	0.000	—	0.000	—	0.000	—	0.000	—	0.000	—	250.000
0.010	7.963 (26)	—	6.119 (24)	—	1.481 (22)	—	2.485 (20)	—	4.609 (18)	—	25.000
0.020	1.157 (13)	—	3.644 (11)	—	5.813 (9)	—	5.366 (7)	—	3.251 (4)	8.0 (5)	12.500
0.030	6.181 (9)	—	1.169 (6)	—	1.074 (4)	1.08 (4)	5.131 (3)	4.70 (3)	1.157 (1)	8.52 (2)	8.333
0.040	1.75 (6)	—	1.812 (4)	1.78 (4)	1.032 (2)	9.2 (3)	2.116 (1)	1.52 (1)	—	8.4 (1)	6.250
0.050	2.842 (5)	2.92 (5)	3.335 (3)	3.08 (3)	1.135 (1)	8.4 (2)	—	6.2 (1)	—	1.8	5.000
0.060	2.350 (4)	2.38 (4)	2.054 (2)	1.78 (2)	2.147 (1)	2.6 (1)	—	1.24	—	3.0	4.167
0.070	1.048 (3)	1.042 (3)	7.037 (2)	5.4 (2)	—	5.4 (1)	—	1.8	—	4.0	3.571
0.080	3.185 (3)	3.08 (3)	1.597 (1)	1.14 (1)	—	8.2 (1)	—	2.4	—	4.8	3.125
0.090	7.458 (3)	6.98 (3)	2.690 (1)	2.0 (1)	—	1.12	—	3.0	—	5.6	2.778
0.100	1.457 (2)	1.32 (2)	2.042 (1)	3.0 (1)	—	1.4	—	3.4	—	6.2	2.500

Table 4. Comparison of the energy levels with the exact ones for the potential $V(x) = -x^2 + Bx^4$; $B > 0$.

	E_0		E_1		E_2		E_3		E_4		
B	WKB	Ref	WKB	Ref	WKB	Ref	WKB	Ref	WKB	Ref	V_{min}
0.001	−248.081	—	−248.081	—	−246.015	—	−246.015	—	−243.166	—	−250.000
0.010[b]	−23.586	−23.596	−23.586	−23.596	−20.820	−20.830	−20.820	−20.830	−18.121	−18.131	−25.000
0.050[b]	−3.626	−3.642	−3.624	−3.640	−1.263	−1.253	−1.078	−1.151	0.191	0.368	−5.000
0.055[a]	−3.219	−3.235	−3.214	−3.231	−0.977	−0.929	−0.640	−0.755	0.522	0.539	−4.545
0.078[a]	−1.874	−1.897	−1.826	−1.859	—	0.009	0.752	0.669	2.030	2.066	−3.205
0.099[a]	−1.277	−1.291	−1.121	−1.184	0.306	0.486	1.524	1.500	3.035	3.050	−2.525
0.100[b]	−1.252	−1.265	−1.089	−1.153	0.333	0.509	1.564	1.543	3.086	—	−2.500
0.130[c]	—	−0.796	—	−0.540	0.942	1.054	2.437	2.437	4.125	4.228	−1.923
0.198[a]	—	−0.319	0.435	0.264	1.938	1.990	3.844	3.854	6.049	6.062	−1.263
0.395[a]	—	0.163	1.424	1.383	3.623	3.660	6.278	6.294	9.283	9.300	−0.633
0.50[c]	0.067	0.294	1.765	1.742	4.220	4.257	7.158	7.175	10.469	10.490	−0.500
0.75[c]	0.256	0.508	2.370	2.370	5.302	5.340	8.772	8.792	12.660	12.680	−0.333
1.0[c]	0.393	0.658	2.818	2.831	6.122	6.162	10.007	10.028	14.345	14.369	−0.250
5.0[c]	1.224	1.594	5.897	5.948	11.941	12.009	18.935	18.967	26.673	26.715	−0.050
10[c]	1.666	1.670	7.668	7.748	15.391	15.40	24.289	24.325	34.115	34.168	−0.025
100[c]	3.933	4.842	17.223	17.40	34.141	34.330	53.544	53.607	74.917	75.070	−0.002
1000	8.625	—	37.384	—	73.822	—	115.730	—	160.509	—	−0.000

[a] Values compared with Somorjai and Hornig (1962).
[b] Values compared with Banerjee and Bhatnagar (1978).
[c] Values compared with those calculated by Oset and Salcedo (1984).

also for even levels, but it is not observed because it occurs in the neighbourhood of the maximum, and in this zone the JWKB energy levels are very inexact because of the large tunnelling effect. For the potential type γ we made a comparison with the values of the energy levels when $B = 1.0$ in figure 5. This is made with the values obtained by Balsa *et al* (1983) using variational methods. In this figure the values are

$$E'_n = E_n + \tfrac{1}{4}A^2$$

for the same reason as before, where $\frac{1}{4}A^2$ is the depth of the potential.

We observe in the last two figures that when the transparency of the barrier between the two wells is large, the JWKB approximation is bad, but this is due to the low order of approximation used.

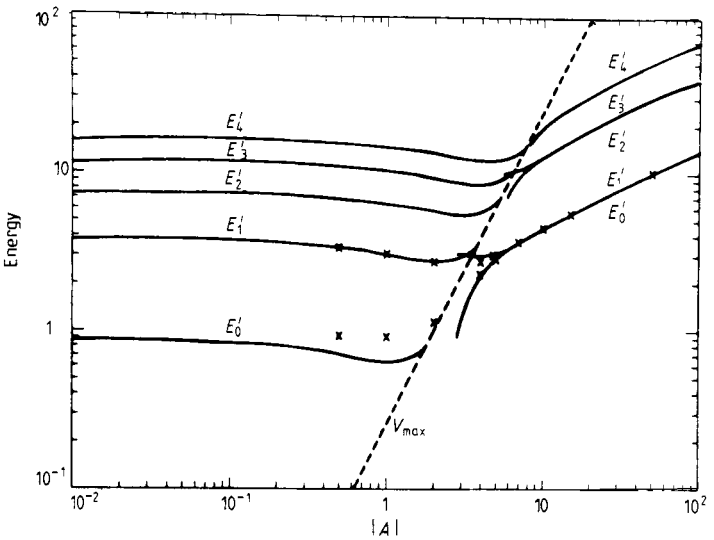

Figure 5. Energy levels of the oscillator $V(x) = -Ax^2 + x^4$. The full curves are our results for the energy levels ($E'_n = E_n + A^2/4$). The crosses are the Balsa *et al* (1983) values. The broken line is the value of the potential maximum ($V_{max} = A^2/4$).

Table 5. Comparison of the splitting of the energy levels with the values obtained with the Oset and Salcedo (1983) program for $A = -1$ and several values of $B(B > 0)$. The index indicates the energy levels taken into consideration.

	$2\Delta E_{0,1}$		$2\Delta E_{2,3}$	
B	WKB	Oset and Salcedo	WKB	Oset and Salcedo
0.010	0.000	0.000	0.000	0.000
0.050	0.002	0.002	0.185	0.102
0.100	1.163	0.112	1.231	1.034
0.50	1.698	1.450	2.938	2.926
1.00	2.425	2.176	3.885	3.874
5.0	4.673	4.363	6.995	6.979
10	6.001	5.548	8.898	8.879
100	13.290	12.598	19.402	19.376
1000	28.758	27.339	41.908	41.859

In table 5, we present the comparison of the splitting of the energy levels ΔE_n with the values obtained with the Oset and Salcedo (1984) program for $A=-1.0$ and several different values of B.

6. Conclusions

We have studied the energy levels of the anharmonic oscillator with all the signs of the coefficients in the symmetrical potential in the JWKB approximation. We have obtained analytic formulae for the different cases, and the agreement with the previously calculated values is good if we take into consideration that our approximation is to lower order. We have found several examples of the very rare exact values known in the literature about the subject.

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Appendix

The complete elliptic integral of the first kind is usually defined (Abramowitz and Stegun 1972)

$$K(m)=\int_0^1 [(1-t^2)(1-mt^2)]^{-1/2}\,dt, \qquad |m|<1.$$

The hypergeometric function is defined as the solution of the Gauss hypergeometric equation (Luke 1969, Lebedev 1972, Carlson 1977, Exton 1978)

$$F(a,b,c,x)=\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)}\frac{x^n}{n!}$$

where Γ is the Euler gamma function

$$\Gamma(x)=\int_0^{\infty} t^{x-1}\,e^{-t}\,dt, \qquad (\mathrm{Re}\,x>0).$$

The beta function is

$$B(a,b)=\Gamma(a)\Gamma(b)/\Gamma(a+b), \qquad (\mathrm{Re}\,a>0, \mathrm{Re}\,b>0).$$

The beta function is also given by the improper integral

$$B(a,b)=\int_0^1 t^{a-1}(1-t)^{b-1}\,dt, \qquad (\mathrm{Re}\,a>0, \mathrm{Re}\,b>0).$$

In order to study the relations between these functions, we calculate the following integral in two different ways.

First method

$$\int_0^1 (1-y^4)^{-1/2}\,dy = \int_0^1 [(1+y^2)(1-y^2)]^{-1/2}\,dy = 2^{-1/2}K(\tfrac{1}{2})$$

using Byrd and Friedman (1954).

Second method: with the change of variables $y^4 = t$, we obtain

$$\int_0^1 (1-y^4)^{-1/2}\,dy = \tfrac{1}{4}\int_0^1 t^{-3/4}(1-t)^{-1/2}\,dt = \tfrac{1}{4}\int_0^1 t^{1/4-1}(1-t)^{1/2-1}\,dt = \tfrac{1}{4}B(\tfrac{1}{4},\tfrac{1}{2})$$

according to the definition of the beta function. Therefore the first relation found between the elliptic integral of the first kind and the beta function is

$$2^{3/2}K(\tfrac{1}{2}) = B(\tfrac{1}{4},\tfrac{1}{2})$$

used in § 5.

The well known relation between the elliptic integral and the hypergeometric function

$$K(m) = (\pi/2)F(\tfrac{1}{2},\tfrac{1}{2},1,m) \qquad |m|<1$$

gives also

$$2^{3/2}K(\tfrac{1}{2}) = 2^{1/2}\pi F(\tfrac{1}{2},\tfrac{1}{2},1,\tfrac{1}{2}) = B(\tfrac{1}{4},\tfrac{1}{2}).$$

References

Abramowitz M and Stegun I A (ed) 1972 *Handbook of Mathematical Functions* (New York: Dover)
Balsa R, Plo M, Esteve J C and Pacheco A F 1983 *Phys. Rev.* D **28** 1945
Banerjee K 1976 *Lett. Math. Phys.* **1** 323
Banerjee K and Bhatnagar S P 1978 *Phys. Rev.* D **18** 4767
Banerjee K, Bhatnagar S P, Choudry V and Kanwal S S 1978 *Proc. R. Soc.* A **360** 575
Bazley M and Fox D 1961 *Phys. Rev.* **124** 483
Bender C M, Olaussen K and Wang P S 1977 *Phys. Rev.* D **16** 1740
Bender C and Wu T T 1969 *Phys. Rev.* **184** 1231
Bhattacharya B, Bhattacharya S and Roy Chowdhury R 1984 *Fizika* **16** 161
Biswas S N, Datta K, Saxena R P, Srivastava P K and Varma V S 1971 *Phys. Rev.* D **4** 3617
—— 1973 *J. Math. Phys.* **14** 1190
Bradbury T C 1968 *Theoretical Mechanics* (New York: Wiley)
Byrd P F and Friedman M D 1954 *Handbook of elliptic integrals for engineers and scientists* (Berlin: Springer)
Carlson B C 1977 *Special functions of applied mathematics* (New York: Academic)
Caswell W E 1979 *Ann. Phys., NY* **123** 153
Cizek J and Vrscay E R 1982 *Int. J. Quantum Chem.* **21** 27
Dias de Deus J 1982 *Phys. Rev.* D **26** 2782
Díaz Bejarano J and Martín Sánchez A 1983 *An. Fis. Ser.* A **79** 8
Díaz Bejarano J, Miró Rodriguez C and Martín Sánchez A 1985 submitted for publication
Díaz Bejarano J, Martín Sánchez A and Miró Rodriguez C 1982 *An. Fis. Ser.* A **78** 159
Drummond J E 1981 *J. Phys. A: Math. Gen.* **14** 1951
Exton H 1978 *Handbook of hypergeometric integrals* (Chichester: Ellis Horwood)
Fernández F M, Arteca G A and Castro E A 1983 *Physica* **122A** 37
Fernández F M and Castro E A 1983 *Phys. Rev.* A **27** 663
Flessas G P 1983 *Phys. Lett.* **95A** 361
Flessas G P, Whitehead R R and Rigas A 1983 *J. Phys. A: Math. Gen.* **16** 85
Galindo A and Pascual P 1978 *Mecánica Cuántica* (Madrid: Alhambra)
Hioe F T and Montroll E W 1975 *J. Math. Phys.* **16** 1945
Hioe F T, MacMillen D and Montroll E W 1978 *Phys. Rep.* **43** 305

Hirsbrunner B 1982 *Helv. Phys. Acta* **55** 295
Killingbeck J 1977 *Rep. Prog. Phys.* **40** 963
Laane J 1970 *Appl. Spectrosc.* **24** 73
Lebedev N N 1972 *Special functions and their applications* (New York: Dover)
Loeffel J J, Martin A, Simon B and Wightman A S 1969 *Phys. Lett.* **30B** 656
Luke Y L 1969 *The special functions and their approximations* (New York: Academic)
Martín Sánchez A 1983 *PhD thesis, Aplicación de las functiones elípticas a los osciladores alineales* Universidad de Extremadura
Marziani M F 1984 *J. Phys. A: Math. Gen.* **17** 547
Mathews P M and Eswaran K 1972 *Lett. Nuovo Cimento* **5** 15
Mathews P M, Seetharaman M, Raghavan S and Bhargava V T A 1981 *Phys. Lett.* **83A** 118
Oset F and Salcedo L L 1984 *An. Fis. Ser.* A **80** 109
Pascual P 1969 *An. Fis. Ser.* A **75** 77
Radmore P M 1980 *J. Phys. A: Math. Gen.* **13** 173
Reid C E 1965 *J. Chem. Phys.* **43** S186
Shepard H K 1983 *Phys. Rev.* D **27** 1288
Somorjai R L and Hornig D F 1962 *J. Chem. Phys.* **36** 1980
Voros A 1983 *Ann. Inst. H Poincaré* **XXXIX** 211